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MOMENTS FOR A MODIFIED MAKEHAM LAW OF MORTALITY. (U)

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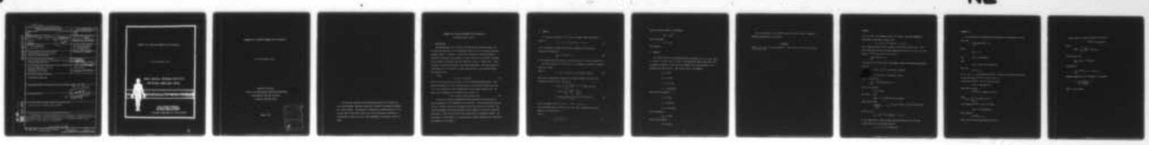

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The moments of a modified Makeham law of mortality are developed into a computational form. General results for the development of moments for survival distributions were used in this development. A numerical example of the evaluation of these moments is presented.

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MOMENTS FOR A MODIFIED MAKEHAM LAW OF MORTALITY

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MOMENTS FOR A MODIFIED MAKEHAM LAW OF MORTALITY

R. Clifton Bailey, Ph.D.

1. INTRODUCTION

The Makeham model for the force of mortality has intuitive appeal and an excellent reputation as an appropriate model for use in actuarial problems (Jordan, [1967]). Recently, I have been interested in survival problems associated with medical procedures for which the force of mortality following the procedure is initially high and eventually falls to a lower rate. Since follow-up is generally limited, long-term extrapolation, which may involve an increasing force of mortality at some longer period, is not modeled beyond the constant term.

I offer

$$h(t) = \delta + \alpha \exp(-\gamma t) \tag{1}$$

as the force of mortality for what I call the modified Makeham model. Note that the parameter δ is the long-term risk, α is an initial excess risk, and γ is a rate constant for the disappearance of the initial excess risk. For this force of mortality the probability of survival is

$$1-F(t) = \exp[-\delta t-(\alpha/\gamma)(1-\exp(-\gamma t))]. \tag{2}$$

In some trial analyses maximum likelihood estimation has been used to obtain estimates of the parameters for this model. The likelihood function was set up to reflect that not all patients fail in the course of the study. The iterative maximum likelihood procedure we used employs a routine for numerical derivatives. This has facilitated the exploration of alternative models. The following is a development of a computationally useful expression for computing the moments for this model.

2. MOMENTS

It is shown in Appendix I that the ν-th moment about the origin is given by

$$\mu'_\nu = \nu\int_0^\infty t^{\nu-1}[1-F(t)]dt, \quad \nu = 1,2,\ldots \tag{3}$$

for a non-negative random variable with cumulative distribution function $F(t)$ provided

$$\lim_{t\to\infty} t^\nu(1-F(t)) = 0. \tag{4}$$

For the modified Makeham model the required limit is shown in Appendix II.

Now these results can be applied to the problem of finding the moments about the origin.

That is,

$$\mu'_\nu = \nu\int_0^\infty t^{\nu-1}\exp[-\delta t-(\alpha/\gamma)(1-\exp(-\gamma t))]dt. \tag{5}$$

Next use the exponential expansion for $\exp[(\alpha/\gamma)\exp(-\gamma t)]$ and do the integration term by term. This gives

$$\begin{aligned}\mu'_\nu &= \nu\exp(-\alpha/\gamma)\sum_{i=0}^{\infty}\frac{(\alpha/\gamma)^i}{i!}\int_0^\infty t^{\nu-1}\exp[-(\delta+i\gamma t]dt\\ &= \nu\exp(-\alpha/\gamma)\sum_{i=0}^{\infty}\frac{(\alpha/\gamma)^i}{i!}\frac{\Gamma(\nu)}{(\delta+i\gamma)^\nu}\\ &= \frac{\Gamma(\nu+1)\exp(-\alpha/\gamma)}{\delta^\nu}\sum_{i=0}^{\infty}(\alpha/\gamma)^i/[i!(1+i\gamma/\delta)^\nu],\end{aligned} \tag{6}$$

where the gamma function $\Gamma(\nu+1)=\nu!$ for $\nu = 1,2,3,\ldots$.

The raw moments given in equation (6) are used to compute the central moments in the usual way.

That is,

$$\mu_\nu = \sum_{i=0}^{\nu}\binom{\nu}{i}\mu'_i\,\mu_1^{\nu-i}. \tag{7}$$

From the central moments the skewness

$$\sqrt{\beta_1} = \mu_3/\mu_2^{1.5}$$

and the kurtosis

$$\beta_2 = \mu_4/\mu_2^2$$

are computed.

3. EXAMPLE

In a trial data set the maximum likelihood estimates for α, γ, and δ were $\hat{\alpha} = 1.388$, $\hat{\gamma} = 3.506$, and $\hat{\delta} = 0.120$. The expressions given above were used to compute estimates of the first four moments, the central moments, the skewness, and the kurtosis. The results are for the moments

$$\hat{\mu}_1' = 5.691,$$

$$\hat{\mu}_2' = 93.526,$$

$$\hat{\mu}_3' = 2{,}337.114,$$

$$\hat{\mu}_4' = 77{,}902.685;$$

for the central moments

$$\hat{\mu}_2' = 61.143,$$

$$\hat{\mu}_3' = 1109.016,$$

$$\hat{\mu}_4' = 39{,}730.361;$$

for the skewness

$$\sqrt{\hat{\beta}_1} = 2.320;$$

and for the kurtosis

$$\hat{\beta}_2 = 10.627.$$

The calculations for the moments were done by Mr. James P. Summe on a FORTRAN program which he developed.

REFERENCE

Jordan, C. W., Jr., Life Contingencies. The Society of Actuaries, Chicago, pp. 20-24, 1967.

APPENDIX I

Let μ'_ν be the ν-th raw moment about the origin. Then some elementary operations can be used to verify that

$$\mu'_\nu = \nu\int_0^\infty t^{\nu-1}\,[1-F(t)]dt,\ \nu = 1,2,...$$

for a random variable t with cumulant distribution fuction F(t). This may be verified directly through integration by parts if we put $u = 1-F(t)$ and $dv = t^{\nu-1}dt$ and if F(t) is such that

$$\lim_{t\to\infty} (1-F(t))t^\nu = 0.$$

The result also follows from a development based on the moment-generating function

$$M(s) = \int_0^\infty e^{st}\,h(t)\exp[-\int_0^t h(x)dx]dt.$$

we set $u = e^{st}$ and $dv = h(t)\exp-\int_0^t h(x)dx$,

here s is such that

$$\lim_{t\to\infty} \quad e^{st}\exp-\int_0^t h(x)dx = 0.$$

That is, we require

$$s < \int_0^t h(x)dx$$

for some t. Then

$$M(s) = 1 + s\int_0^\infty e^{st}\exp[-\int_0^t h(x)dx]dt.$$

From this we find

$$\frac{d^\nu M(s)}{d\,s^\nu} = \nu\int_0^\infty t^{\nu-1}e^{st}(1-F(t)) + s\int_0^\infty t^\nu e^{st}(1-F(t))dt$$

or for s = 0,

$$\mu'_\nu = \nu\int_0^\infty t^{\nu-1}[1-F(t)]dt\ \ \nu = 1,2,....$$

In the same way the central moment-generating function can be used to show that the ν-th central moment is

$$\mu_\nu = \nu\int (t-\mu_1)^{\nu-1}(1-F(t))dt.$$

APPENDIX II

The theorem for the moments can be applied to our problem if we can show that

$$L = \lim_{t\to\infty} (N(t)/D(t)) = 0,$$

where

$$N(t) = t^{\nu},$$

and $$D(t) = Q(t)\exp(\delta t),$$

when

$$Q(t) = \exp[(\alpha/\gamma)(1-\exp(-\gamma t))].$$

Note that

$$N^{(\nu)}(t) = \nu!$$

for $\nu = 1,2,\ldots.$ Also observe that $Q(\infty) = \exp(\alpha/\gamma)$ and that by induction we can show that $Q^{(i)}(\infty) = 0$ for $i = 1,2,\ldots.$

To show that $Q^{(i)}(\infty) = 0$, we first note that

$$Q'(t) = \alpha Q(t)\exp(-\gamma t).$$

From this we obtain

$$Q^{(i+1)}(t) = \alpha\exp(-\gamma t)\ S_i(t),$$

where
$$S_i(t) = \sum_{j=0}^{i} \binom{i}{j} Q^{(j)}(t)(-\gamma)^{i-j}.$$

If we assume $Q^{(i)}(\infty) = 0$ for $i>1$, then

$$S_i(\infty) = \alpha(-\gamma)^i\exp(\alpha/\gamma),$$

which implies

$$Q^{(i+1)}(\infty) = 0,$$

since $S_i(\infty)$ is finite and $\exp(-\delta t)\to 0$ as $t\to\infty$.

Now a similar argument applied to $D(t)$ gives

$$D^{(i)}(t) = R_i(t)\exp\delta t,$$

where

$$R_i(t) = \sum_{j=0}^{i} \binom{i}{j} \delta^{i} = {}^{j}{}_{Q}{}^{(j)}(t),$$

from which we have

$$\lim_{t\to\infty} R_i(t) = \delta^i \exp(\alpha/\gamma).$$

Consequently,

$$\lim_{t\to\infty} D^{(i)}(t) = \infty,\ i = 0,1,2....$$

Repeated application of L'Hospital's rule gives

$$L = \lim_{t\to\infty} \frac{N^{(\nu)}(t)}{D^{(\nu)}(t)} = 0$$

when ν is an integer.